

Gamow's Theory of Alpha Decay

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Abstract

In 1928, George Gamow used some knowledge about tunneling and the WKB approximation (to the one-dimensional time independent Schrodinger equation) to provide the first theoretical account of alpha decay (emission of 2 protons and 2 neutrons, ie, a positive charged particle, $2e$, by heavy nuclei). In this paper, it will be discussed, in first place, the WKB approximation and its application to one-dimensional potentials which act as barriers (the scattering problem). After that, the alpha decay will be discussed.

Introduction: WKB approximation

One-dimensional time independent Schrodinger equation is often intractable. Suppose a particle with energy E moving through a region where $V(x)$ is constant. If $E > V(x)$, the wave function is of the form:

$$\psi(x) = Ae^{\pm ikx}, k = \frac{\sqrt{2m(E-V)}}{\hbar} \quad (1)$$

Now, suppose that V is not constant, but varies rather slowly in comparison to his wavelength. So, we can say that the potential is essentially constant and $\psi(x)$ remains with a sinusoidal form. Nevertheless, his wavelength and amplitude change slowly with x .

In the same way, if $E < V(x)$ (and V is constant), then the wave function is a real exponential: $ik \rightarrow \kappa = \frac{\sqrt{2m(V-E)}}{\hbar}$. If $V(x)$ is not constant, but varies slowly in comparison with $\frac{1}{\kappa}$, the solution remains practically exponential.

If we write the time independent Schrodinger equation as:

$$\frac{d^2\psi}{dx^2} = -\frac{\sqrt{2m(E-V(x))}}{\hbar^2}\psi = -\frac{p^2(x)}{\hbar^2}\psi \quad (2)$$

and if we assume that ψ is a complex function, then:

$$\psi(x) = A(x)e^{i\phi(x)} \quad (3)$$

$$\frac{d\psi}{dx} = \left(\frac{dA}{dx} + iA\frac{d\phi}{dx}\right)e^{i\phi(x)} \quad (4)$$

$$\frac{d^2\psi}{dx^2} = \left(\frac{d^2A}{dx^2} + iA\frac{d^2\phi}{dx^2} + 2i\frac{dA}{dx}\frac{d\phi}{dx} - A\left(\frac{d\phi}{dx}\right)^2\right)e^{i\phi(x)} \quad (5)$$

Putting (5) into equation (2), we conclude that:

$$\frac{d^2A}{dx^2} = A\left[\left(\frac{d\phi}{dx}\right)^2 - \frac{p^2}{\hbar^2}\right] \quad (6)$$

$$\frac{d(A^2\frac{d\phi}{dx})}{dx} = 0 \quad (7)$$

7th equation is easily solved, and:

$$A = \frac{C}{\sqrt{\frac{d\phi}{dx}}} \quad (8)$$

where C is a real constant. 6th equation is not too easy and we need the approximation we have talked

about: the amplitude A varies slowly, so $\frac{d^2A}{dx^2} \ll \left(\frac{d\phi}{dx}\right)^2 - \left(\frac{p}{\hbar}\right)^2$ and then:

$$\left(\frac{d\phi}{dx}\right)^2 = \left(\frac{p}{\hbar}\right)^2, \text{ or } \frac{d\phi}{dx} = \pm \frac{p}{\hbar} \quad (9)$$

$$\phi(x) = \pm \frac{1}{\hbar} \int p(x)dx \quad (10)$$

Using equations (9), (8), (10) and (3), it follows that:

$$\psi(x) \simeq \frac{D}{\sqrt{p(x)}} \exp\left\{\pm \frac{i}{\hbar} \int p(x)dx\right\} \quad (11)$$

$$|\psi(x)|^2 \approx \frac{|D|^2}{p(x)} \quad (12)$$

12th equation represents the probability of finding the particle at point x.

It was assumed that $E > V$, so $p(x)$ is real. If $E < V$, $p(x)$ is imaginary and:

$$\psi(x) \approx \frac{D}{\sqrt{|p(x)|}} \exp\left\{\pm \frac{1}{\hbar} \int |p(x)| dx\right\} \quad (13)$$

Suppose, now, a barrier with a bumpy top, limited to the range (0, a) (out of the barrier, the potential is null). So, to the left of barrier ($x < 0$), the wave function is:

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad (14)$$

k is given in eq.(1), with $V=0$. A and B are the amplitudes of the incident and reflected waves, respectively. To the right of the barrier ($x > a$):

$$\psi(x) = Fe^{ikx} \quad (15)$$

where F is his transmitted amplitude. As we know, the tunneling probability is:

$$T = \left(\frac{|F|}{|A|}\right)^2 \quad (16)$$

In the tunneling region, using eq.(13), we have:

$$\psi(x) \approx \frac{C}{\sqrt{|p(x)|}} \exp\left\{\frac{1}{\hbar} \int_0^x |p(y)| dy\right\} + \frac{D}{\sqrt{|p(x)|}} \exp\left\{-\frac{1}{\hbar} \int_0^x |p(y)| dy\right\} \quad (17)$$

If the barrier is infinitely broad, we assume $C=0$, and:

$$\frac{|F|}{|A|} \sim \exp\left\{-\frac{1}{\hbar} \int_0^a |p(y)| dy\right\} \quad (18)$$

Eq.(16) and (18) allow us to write:

$$T = \left(\frac{|F|}{|A|}\right)^2 = \exp\left\{-\frac{2}{\hbar} \int_0^a |p(y)| dy\right\} = \exp\{-2\gamma\} \quad (19)$$

The alpha decay

The nucleons (protons and neutrons) in a nucleus are bound together - their total energy is less than the total energy of the separated particles. The

binding energy is the amount of energy given up when the nucleus is formed. Plotting the binding energy per nucleon versus the mass number (A) (Figure 1) shows that starting from Hydrogen, nuclei become more stable as there are more protons and neutrons, until Iron. After that, the trend reverses.

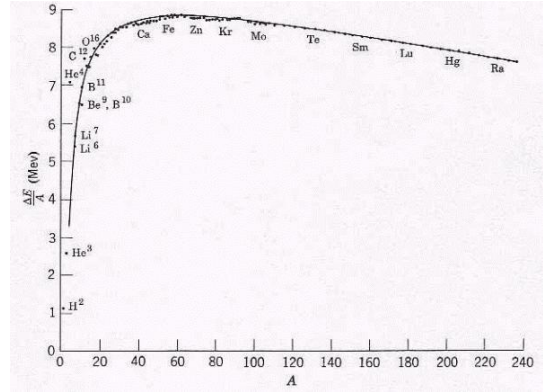


Figure 1. Nuclear Binding Energy. [8]

Figure 2 shows the distribution of the stable nuclei.

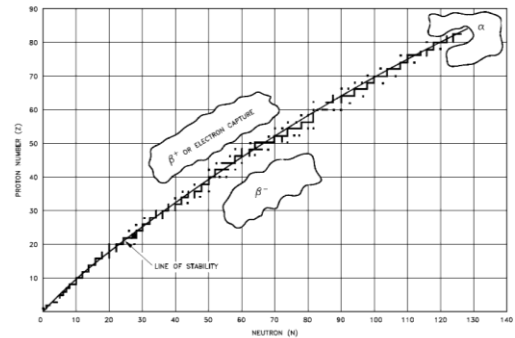


Figure 2. Neutron (x axis)/Proton (y axis) ratio and decay. Notice that this ratio in stable isotopes becomes greater than 1 as the mass increases. [7]

As the mass numbers become higher, the ratio of neutrons to protons in the nucleus becomes larger. There are no stable nuclei with a mass number higher than 83 or a neutron number higher than 126. Notice how the stability band pulls away from the $P=N$ line. Figure 2 also shows all the trends of decay. There are some exceptions to the trends but generally a nucleus will decay following the trends (in multiple steps) until it becomes stable. This process is called a radioactive series. The curve of stable nuclei portrays in Figure 2 is the result of the balancing act between the various repulsive and attractive effects: electric force, uncertainty principle, Pauli's exclusion principle, strong interaction and number of neutrons.

Unstable nuclei, called radioactive isotopes, will undergo nuclear decay to become more stable. There are only certain types of nuclear decay which means that most isotopes can't jump directly from being unstable to being stable. It often takes several decays to eventually become a stable nucleus. When unstable nuclei decay, the reactions generally involve the emission of a particle and or energy. Half-lives are characteristic properties of the various unstable atomic nuclei and the particular way in which they decay. Alpha and beta decay are generally slower processes than gamma decay. Half-lives for beta decay range upward from 10^{-2} sec and, for alpha decay, upward from about 10^{-6} sec. Half-lives for gamma decay may be too short to measure ($\sim 10^{-14}$ second). There are 5 types of nuclear decay: α , β , γ , β^+ and Electron Capture (EC).

The strong force, despite its strength, has a very short range; it can't even reach from one end of a fair-sized atomic nucleus to the other. If a proton is at the edge of a big nucleus, it can feel the pulling strong force only from the particles in the neighborhood, but there is an electromagnetic force, which tends to push it out, all the way from the other side of the nucleus. There is a sensitive balance between these two competing forces. The nucleus needs not to acquire extra energy to escape; the quantum mechanical effect called tunneling allows a certain probability of escape through a potential wall. α -decay, in which just a small chunk breaks off from the main nucleus, is a rather mild case of fission; in more dramatic examples, the nucleus can break more or less in half. The broken-off chunk most often is packed into a helium nucleus (α particle) because it is in a more stable form. Figure 3 shows the effect of tunneling through the Coulomb barrier; the nucleus has a small probability of escape to the outside depending on the height and width of the wall, and this probability is related to the transmission coefficient T, as we will see.

The alpha decay can be modeled by eq.(20):



Due to conservation laws, α particle must have $Z=2$ and $A=4$, ie, an Helium nucleus. Reaction (20) is only possible if that particle suffers tunneling effect through the Coulomb potential barrier (between Y and alpha particles). The alpha particle is not in a bound state, otherwise, this decay could not occur. Moreover, alpha energy is positive and its escape is only inhibited by the barrier presence.

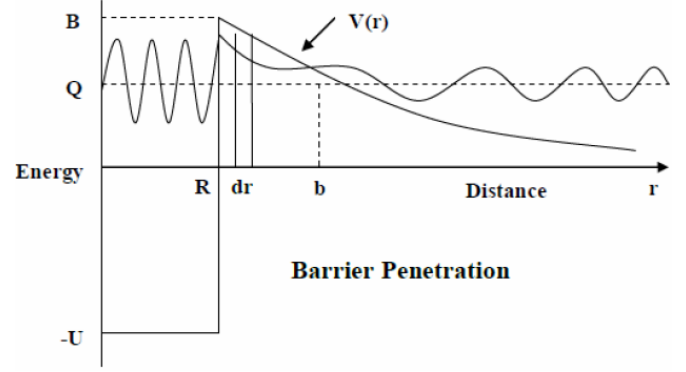


Figure 3. Gamow's model for the potential energy of an alpha particle in a radioactive nucleus. Notice that R is the radius of the nucleus, Q is the average energy of the alpha particle, $B=V(R)$ [$V(r)$ is the Coulomb potential] and b is supposed to be the turning point, where $V(b)=Q$. For $r < R$, there is just the result of the balancing act between the various repulsive and attractive effects, and nucleus's potential energy is $-U$ (nuclear binding). [5]

The potential energy of this problem can be modeled by eq.(21).

$$V(r) = \begin{cases} \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r}, & r \geq R \\ -U, & r < R \end{cases} \quad (21)$$

If we assume that the potential is spherical (with a central potential $V(r)$), the wavefunction is $\Psi(r) = R(r)\Theta(\theta)\Phi(\varphi)$. The eigenfunctions of the angular part are the spherical harmonic functions: $Y_{l,m_l}(\cos\theta, \varphi)$. Then, the potential given by eq.(21) is added to a centrifugal potential $\frac{l(l+1)\hbar^2}{2mr^2}$. For simplicity, assume that $l=0$, and work with eq.(21). So, using eq. (21) and (19), the Gamow factor is given by eq.(22).

$$\gamma = \frac{1}{\hbar} \int_R^b \sqrt{2m \left(\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r} - Q \right)} dr \quad (22)$$

If we make $Q = V(b)$, so $b = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 Q}$ and $\frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 r} = \frac{Qb}{r}$, then eq.(22) becomes:

$$\gamma = \frac{1}{\hbar} \int_R^b \sqrt{2mQ \left(\frac{b}{r} - 1 \right)} dr = \frac{\sqrt{2mQ}}{\hbar} \int_R^b \sqrt{\frac{b}{r} - 1} dr \quad (23)$$

Mathematica software returns the result of the following integral:

$$\int_u^{u'} \sqrt{\frac{1}{r} - \frac{1}{u'}} dr = \sqrt{u'} \left[\cos^{-1} \sqrt{\frac{u}{u'}} - \sqrt{\frac{u}{u'}} - \left(\frac{u}{u'} \right)^2 \right] \quad (24)$$

Then, eq.(23) becomes:

$$\gamma = \frac{\sqrt{2m} Z_1 Z_2 e^2}{\hbar \sqrt{Q} 4\pi\epsilon_0} g\left(\frac{R}{b}\right) \quad (25)$$

where $g(x) = \arccos \sqrt{x} - \sqrt{x(1-x)}$.

As $Q = mv^2/2$ and $Q = (m_{parent} - m_{daughter} - m_\alpha)c^2$, it is simple to conclude that $\left(\frac{v}{c}\right)^2 = 2\left(\frac{m_{parent}}{m_\alpha} - \frac{m_{daughter}}{m_\alpha} - 1\right)$. For example, in U-238 case, $v_\alpha \cong 1.44 \times 10^7$ m/s and $v/c \sim 0.05$ (ie, v is 5% of c).

Now, it is clear that the probability of alpha decay is given by eq.(26).

$$P(\alpha) = \lambda \cong \frac{vT}{R} \quad (26)$$

In last equation, v is the speed of the alpha particle (obtained by the kinetic energy expression $Q = mv^2/2$), T its transmission coefficient and R the radius of the nucleus. Indeed, the constant of disintegration is the product of the probability of getting through the barrier (T) by the number of attempts to make the particle go through it (given by the number of collisions with the surface in unit time, which is approximately equal to v/R). Using eq.(19), (25) and (26), then:

$$\lambda \cong \frac{1}{R} \sqrt{\frac{2Q}{m}} \exp\left\{-\frac{2\sqrt{2m}}{\hbar} \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0} g\left(\frac{R}{b}\right)\right\} \quad (27)$$

$$\ln \lambda = W + U \frac{1}{\sqrt{Q}} \quad (28)$$

$$W = \frac{1}{2} \ln\left(\frac{2Q}{m}\right) - \ln R \quad (29)$$

$$U = -\frac{2}{\hbar} \sqrt{2m} \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0} g\left(\frac{R}{b}\right) \quad (30)$$

We know that $N = N_0 e^{-\lambda t}$ and, if $N/N_0 = 1/2$, then $\frac{\ln 2}{\lambda} = \tau_{1/2}$. By the way, notice that due this result and the equations (26) and (19), it is obvious that

$$\tau_{1/2} = \frac{R \ln 2}{v} e^{2\gamma}$$

$$\tau_{1/2} = \frac{\ln 2}{\frac{1}{R} \sqrt{\frac{2Q}{m}}} \exp\left\{\frac{2\sqrt{2m}}{\hbar} \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0} g\left(\frac{R}{b}\right)\right\} \quad (31)$$

$$\ln \tau_{1/2} = W' + U' \frac{1}{\sqrt{Q}} \quad (32)$$

$$W' = \frac{1}{2} \ln\left(\frac{m}{2Q}\right) + \ln(R \ln 2) \quad (33)$$

$$U' = -U \quad (34)$$

[W' and U' are not the derivatives of W and U].

On the other hand, if we want to know the variation of $\ln \lambda$ (or $\ln(\ln 2 / \tau_{1/2})$) vs Z , it is obvious that $Z = Z_1 + Z_2 = Z_1 + 2$, and then we obtain:

$$\ln\left(\frac{\ln 2}{\tau_{1/2}}\right) = W'' + U'' Z \quad (35)$$

$$W'' = W + \frac{8\sqrt{2m}}{\hbar} \frac{e^2}{4\pi\epsilon_0} g\left(\frac{R}{b}\right) \quad (36)$$

$$U'' = -\frac{4}{\hbar} \sqrt{\frac{2m}{Q}} \left(\frac{e^2}{4\pi\epsilon_0}\right) g\left(\frac{R}{b}\right) \quad (37)$$

Equations (28) and (32) represent linear functions of $\ln \lambda$ (or $\ln \tau_{1/2}$) vs $\frac{1}{\sqrt{Q}}$, where the ‘constants’ are both functions of Q . Indeed, W is a portion of slow variation of Q (the same about W'), and U contain the function $g(R/b)$. Notice that $R/b = Q/B$, where $B = V(R)$, ie, the Coulomb potential value at $r = R$. But $g(R/b)$ is a portion of slow variation of Q , too.

A comparison of the Geiger–Nuttall relation (rule that relates the decay constant of a radioactive isotope with the energy of the alpha particles emitted) with experimental data for different families of nuclei is shown in Figure 4. In this plot, the straight lines confirm the exponential dependence of half-life on alpha energy.

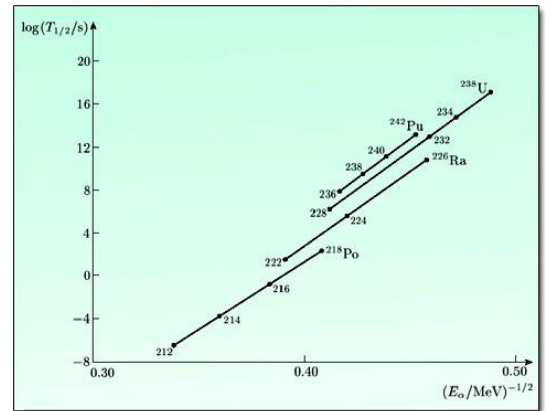


Figure 4. A comparison of the Geiger–Nuttall relation with experimental data for different families of nuclei. [6]

Suppose that we want to calculate the lifetime of the U-238 nuclei. Assuming that the density of nuclear matter is relatively constant, so R^3 is proportional to A , or (empirically) $R=1.07A^{1/3}$ (fm) [$1\text{fm}=10^{-15}\text{m}$]. The energy of the emitted alpha particle is determined by $E = (m_{\text{parent}} - m_{\text{daughter}} - m_{\alpha})c^2$. With the results above we can calculate the lifetime of this species: $\tau_{1/2,U-238} = 2.4 \times 10^{14}$ yrs. But the experimental value for this lifetime is 4.468×10^9 yrs, ie, the results obtained are way off. Some experimental values for lifetimes of U ($Z=90$) are listed in Table 1 (all energies in MeV). If we plot the values of $\ln \tau_{1/2}$ as function of $\frac{1}{\sqrt{Q}}$, we obtain the equation $\ln \tau_{1/2} = -120.2 + 327.19 \frac{1}{\sqrt{Q}}$, that's it, $U'=327.19$ and $W' = -120.2$, on eq. (32).

A	E	τ
238	4.198	4.468×10^9 yr
236	4.494	2.342×10^7 yr
234	4.775	2.455×10^5 yr
232	5.320	68.9 yr
230	5.888	20.8 day
228	6.680	9.1 min
226	7.570	0.35 s

Table 1. Some experimental values for lifetimes of U, $Z=90$; energies in MeV. [4]

Note the extraordinary sensitivity to nuclear masses: a tiny change in E produces an enormous change in the lifetime.

Why these results? In first place, we must see that we have assumed $l=0$, i.e., we have neglected the centrifugal potential term. This term has an effect of increasing the height and width of the Coulomb barrier.

Note that this problem could be solved, in first approximation, knowing the solution of the potential barrier (Figure 5). Indeed, looking for Figure 6, any potential can be modeled by the juxtaposition of some square potential barriers. Let us explain this point of view.

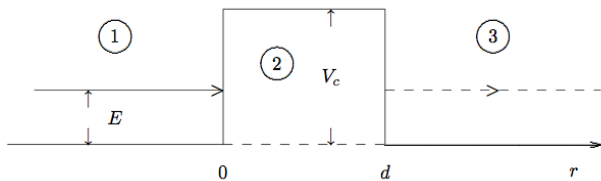


Figure 5. The square potential barrier: V_0 is the the barrier height.

For this problem, it is simple to show that in region 2, $k_2 = \frac{\sqrt{2m(V_0-E)}}{\hbar}$, and $T = [1 + \xi \sinh^2 k_2 d]^{-1}$, where ξ is a constant. If $k_2 d \gg 1$, $\sinh^2 k_2 d \cong \frac{1}{4} e^{2k_2 d}$ and $T \cong \frac{4}{\xi} e^{-2k_2 d}$, or,

$$T \propto e^{-2k_2 d} = \exp\left\{-\frac{2}{\hbar} \sqrt{2m(V_0 - E)} d\right\} \quad (38)$$

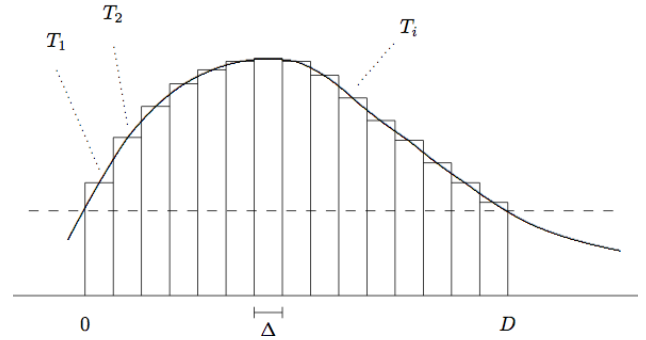


Figure 6. Approximation of a potential by a several square potential barriers, of width Δ . [4]

$$T \cong C \prod_i \exp\left\{-\frac{2}{\hbar} \sqrt{2m(V_i - E)} \Delta\right\} = C \exp\left\{\sum_i -\frac{2}{\hbar} \sqrt{2m(V_i - E)} \Delta\right\} \quad (39)$$

In eq.(39), C is a constant. If $\Delta \rightarrow 0$, then eq.(39) becomes:

$$T \cong C \exp \int dr \left(-\frac{2}{\hbar} \sqrt{2m(V - E)}\right) \quad (40)$$

Using WKB approximation, we see that $C \cong 1$. Figure 7 represents the application of this procedure to the alpha decay.

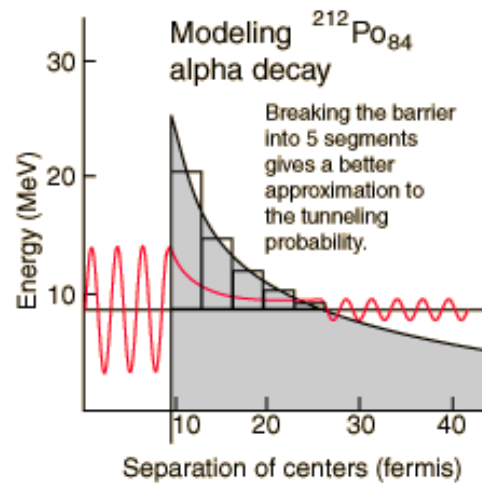


Figure 7. Approximation of a smooth barrier by a juxtaposition of square potential barriers. [8]

Furthermore, known the expression for Gamow's factor (see eq.(25)), we can approach $\arccos(x)$ if $x \ll 1$. Indeed, that equation can be written as follows:

$$\gamma = \frac{\sqrt{2mQ}}{\hbar} \left[b \arccos \sqrt{\frac{R}{b}} - \sqrt{R(b-R)} \right] \quad (41)$$

For $x \ll 1$, $\arccos x \cong \frac{\pi}{2} - x$. By chance, usually $b \gg R$, and then, $\arccos \sqrt{\frac{R}{b}} \cong \frac{\pi}{2} - \sqrt{\frac{R}{b}}$, and $\sqrt{R(b-R)} \cong \sqrt{Rb}$. In this case, eq.(41) is approximated by eq.(42).

$$\gamma \cong \frac{\sqrt{2mQ}}{\hbar} \left(\frac{\pi}{2} b - 2\sqrt{Rb} \right) \quad (42)$$

It is simple to show (and we need some auxiliary and linear calculations) that:

$$\gamma \cong K \frac{Z_1}{\sqrt{Q}} - J \sqrt{RZ_1} \quad (43)$$

$$K = \left(\frac{e^2}{4\pi\epsilon_0} \right) \frac{\pi\sqrt{2m}}{\hbar} \cong 1.980 MeV^{\frac{1}{2}} \quad (44)$$

$$J = \left(\frac{e^2}{4\pi\epsilon_0} \right)^{\frac{1}{2}} \frac{4\sqrt{m}}{\hbar} \cong 1.485 fm^{-\frac{1}{2}} \quad (45)$$

There has been the fact that previous findings which relate lambda (or half-life) with the energy and Z, could be obtained using this last term, which is nothing more than an approach of the Gamow's theory of Geiger-Nuttall equation.

Using last approximation, we can obtain the Gamow's factor for U-238 (for example), which results in 49.9. For Po-212, this factor is 20.4.

Conclusions and discussion

The success of the theoretical explanation of the Geiger-Nuttall relation using the tunneling probability is one of the first experimental confirmations of quantum mechanics. If we take the results as satisfactory, we conclude that the nucleus can be treated as a well potential for radial distances smaller than the radius of the core and a Coulomb potential for distances larger than the nuclear radius. The model also explains the decrease in the probability of decay to states of higher levels, which have a centrifugal potential term, dependent on the particle angular momentum, which results in an effective potential barrier taller and wider. This result also confirms the predictions of quantum mechanics.

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