Chapter 15

β Decay

Note to students and other readers: This Chapter is intended to supplement Chapter 9 of Krane's excellent book, "Introductory Nuclear Physics". Kindly read the relevant sections in Krane's book first. This reading is supplementary to that, and the subsection ordering will mirror that of Krane's, at least until further notice.

β-particle's are either electrons[1] or positrons that are emitted through a certain class of nuclear decay associated with the "weak interaction". The discoverer of electrons was Henri Becquerel, who noticed that photographic plates, covered in black paper, stored near radioactive sources, became fogged. The black paper (meant to keep the plates unexposed) was thick enough to stop α-particles, and Becquerel concluded that fogging was caused by a new form of radiation, one more penetrating than α-particles. The name "β", followed naturally as the next letter in the Greek alphabet after α, α-particles having already been discovered and named by Rutherford.

Since that discovery, we have learned that β-particles are about 100 times more penetrating than α-particles, and are spin-$\frac{1}{2}$ fermions. Associated with the electrons is a conserved quantity, expressed as the quantum number known as the *lepton number*. The lepton number of the negatron is, by convention +1. The lepton number of the positron, also the anti-particle of the negatron, is -1. Thus, in a negatron-positron annihilation event, the next lepton number is zero. Only leptons can carry lepton number. (More on this soon.) Recall, from Chapter 13 (Chapter 6 in Krane), our discussion of the various decay modes that are associated with β decay:

[1] Technically, the word "electron" can represent either a negatron (a fancy word for e^-) or a positron (e^+). I'll use "electron" interchangeably with this meaning, and also e^-. Usually the context determines the meaning.

$$\begin{array}{llll}
{}^A_Z X_N & \longrightarrow & {}^{\ A}_{Z+1}X'_{N-1} + e^- + \overline{\nu}_e & \beta^- \text{ decay} \\
{}^A_Z X_N & \longrightarrow & {}^{\ A}_{Z-1}X'_{N+1} + e^+ + \nu_e & \beta^+ \text{ decay} \\
{}^A_Z X_N & \longrightarrow & {}^{\ A}_{Z-1}X'_{N+1} + \nu_e & \text{electron capture } (\varepsilon)
\end{array} \tag{15.1}$$

We see from the above processes that there are other particles called *neutrinos*. Neutrinos are also spin-$\frac{1}{2}$ leptons (part of the larger fermion family). They are very nearly massless (but proven to have mass[2]). The electron neutrino is given the symbol ν_e, and has lepton number +1. The antineutrino, the $\overline{\nu}_e$, has lepton number -1. A sketch of the organization of fundamental particles is given in Figure 15.1.

Figure 15.1: The particle physics classification of bosons and fermions, with the sub-classifications of baryons and fermions shown.

Three views of β decay

There are three ways of viewing β decay. The first is the "radiological physics view" expressed by (15.1). The next is the "nuclear physics view", where we recognize that the decays of the nuclei are actually caused by transformations of the nucleon constituents, as expressed in (15.2).

$$\begin{array}{rcll}
n & \longrightarrow & p + e^- + \overline{\nu}_e & \beta^- \text{ decay} \\
p & \longrightarrow & n + e^+ + \nu_e & \beta^+ \text{ decay} \\
p + e^- & \longrightarrow & n + \nu_e & \text{electron capture } (\varepsilon)
\end{array} \tag{15.2}$$

A free neutron will decay with a meanlife, $\tau = 885.7(8)$s, about 11 minutes. A free proton is basically stable. Once these nucleons are bound in a nucleus, however, conservation of energy, with the availability of lower energy states, dictates whether or not these processes are free to proceed.

Then, there is the more microscopic view, the "particle physics view" expressed in (15.3),

$$\begin{array}{rcll}
d & \longrightarrow & u + e^- + \overline{\nu}_e & \beta^- \text{ decay} \\
u & \longrightarrow & d + e^+ + \nu_e & \beta^+ \text{ decay} \\
u + e^- & \longrightarrow & d + \nu_e & \text{electron capture } (\varepsilon)
\end{array} \tag{15.3}$$

[2] A direct measurement of neutrino mass suggests that its upper limit is $m_{\nu_e} < 2.2$eV. Indirect measurement of the neutrino mass suggest that $0.04\text{eV} < m_{\nu_e} < 0.3$eV. For the more massive lepton family groups, $m_{\nu_\mu} < 180$keV, and $m_{\nu_\tau} < 15.5$MeV.

that represents the transitions of nucleons, as really transitions between the up (u) and down (d) quarks. A particle physics picture of β^--decay is given in Figure 15.2.

Figure 15.2: The particle physics view of β^--decay. In this case, the weak force is carried by the intermediate vector boson, the W^-. In the case of β^--decay, the weak force is carried by the intermediate vector boson, the W^+, the antiparticle to the W^-. There is also a neutral intermediate vector boson, Z^0, that is responsible for such things as $\nu\nu$ scattering.

Consequences of β-decay's 3-body final state

$\beta^\pm$-decay has 3 "bodies" in the final state: the recoil daughter nucleus, the $e^\pm$, and a neutrino. Typically, the daughter nucleus (even in the case of free neutron decay, is much more massive than the leptons, therefore, the leptons carry off most of the energy. (Even in the worst possible case, that of free neutron decay, the recoil proton can at most about 0.4 keV, or about 0.05% of the reaction Q-value.) Consequently, if one measures the kinetic energy of the resultant electron, one measures a distribution of energies, that (generally) peaks at small energies, and reaches an "end-point" energy, the so-called *β-endpoint*. This β-endpoint represents the case where the ν's energy approaches zero. See Figure 15.3.

Figure 15.3: A typical electron energy spectrum that is measured in a β decay. The endpoint energy is the maximum energy that can be given to the electron, and that is closely related to the reaction Q-value (small recoil correction). At lesser energies, the ν carries off some of the available kinetic energy that Q provides.

This leads naturally to a discussion of ...

15.1 Energy release in β decay

Neutron decay

$$\begin{aligned}
n &\longrightarrow p + e^- + \overline{\nu}_e \\
m_n c^2 &= m_p c^2 + m_{e^-} c^2 + m_{\overline{\nu}_e} c^2 + Q_n \\
Q_n &= m_n c^2 - m_p c^2 - m_{e^-} c^2 - m_{\overline{\nu}_e} c^2 \\
Q_n &= (939.565580(81) - 938.272013(23) - 0.5110999(0))[\mathrm{MeV}] - m_{\overline{\nu}_e} c^2 \\
Q_n &= 0.782568(84)[\mathrm{MeV}] - m_{\overline{\nu}_e} c^2 \qquad (15.4)
\end{aligned}$$

Since $4 \times 10^{-8} < m_{\overline{\nu}_e} c^2 < 2.2 \times 10^{-6}$[MeV], we can safely ignore the neutrino rest mass energy, within the experimental uncertain of the reaction Q,

$$Q_n = 0.782568(84)[\mathrm{MeV}] \tag{15.5}$$

Accounting for proton recoil, the exact relationship between the electron endpoint energy and Q, is given by:

$$\begin{aligned} T_e^{\max} &= (m_p + m_e)c^2 \left[-1 + \sqrt{1 + \frac{2Q_n m_p c^2}{[(m_p + m_e)c^2]^2}} \right] \\ T_e^{\max} &\approx \frac{Q_n}{1 + m_e/m_p} \ . \end{aligned} \tag{15.6}$$

Putting in numerical values, was calculate $T_e^{\max} = 0.782142(84)[\mathrm{MeV}]$, which agrees with the direct measurement of $T_e^{\max} = 0.782(13)[\mathrm{MeV}]$.

We can calculate the proton's recoil energy by using Conservation of Energy:

$$\begin{aligned} T_p^{\max} &= Q_n - T_e^{\max} \\ T_e^{\max} &\approx Q_n \left(1 - \frac{1}{1 + m_e/m_p} \right) \\ T_e^{\max} &\approx Q_n (m_e/m_p) \ . \end{aligned} \tag{15.7}$$

This evaluates numerically to $T_p^{\max} \approx 0.426(84)[\mathrm{keV}]$.

Q for β^--decay

For β^--decay

$${}^{A}_{Z}X_N \longrightarrow {}^{\;\;A}_{Z+1}X'_{N-1} + e^- + \overline{\nu}_e \tag{15.8}$$

Going back to the definition of Q in terms of nuclear masses, and ignoring, henceforth, the mass of the neutrino:

$$Q_{\beta^-} = \left[m_{\mathrm{N}}({}^{A}_{Z}X_N) - m_{\mathrm{N}}({}^{\;\;A}_{Z+1}X'_{N-1}) - m_e \right] c^2 \ , \tag{15.9}$$

where the subscript "N" connotes nuclear (not atomic) masses.

The relationship between the nuclear (no subscript "N") and atomic mass is:

$$m({}^A_Z X_N)c^2 = m_{\rm N}({}^A_Z X_N)c^2 + Z m_e c^2 - \sum_{i=1}^{Z} B_i \ , \tag{15.10}$$

where B_i is the binding energy of the i'th atomic electron.

Substituting (15.10) in (15.9), to eliminate the (less well known) nuclear masses results in:

$$\begin{aligned}
Q_{\beta^-} &= \left[m({}^A_Z X_N) - Z m_e\right] c^2 - \left[m({}_{Z+1}^{\ \ A}X'_{N-1}) - (Z+1)m_e\right] c^2 - m_e c^2 + \left[\sum_{i=1}^{Z} B_i - \sum_{i=1}^{Z+1} B'_i\right] \\
&= \left[m({}^A_Z X_N) - m({}_{Z+1}^{\ \ A}X'_{N-1})\right] c^2 + \left[\sum_{i=1}^{Z} B_i - \sum_{i=1}^{Z+1} B'_i\right] \ , \\
&= \left[m({}^A_Z X_N) - m({}_{Z+1}^{\ \ A}X'_{N-1})\right] c^2 + \left[\sum_{i=1}^{Z} (B_i - B'_i) - B'_{Z+1}\right] \ ,
\end{aligned} \tag{15.11}$$

noting that the electron masses have canceled in this case. The factor

$$\sum_{i=1}^{Z} B_i - \sum_{i=1}^{Z+1} B'_i = \sum_{i=1}^{Z} (B_i - B'_i) - B'_{Z+1}$$

is the difference in the energy of the electronic orbital configuration of the parent and daughter nuclei. Generally, this difference can be ignored. However, in the case of large Z nuclei, it can amount to about 10 keV. For accurate determinations of Q, the difference in atomic electron binding energy must be accounted for.

Q for β^+-decay

Similar considerations for β^+ decay lead to:

$$Q_{\beta^+} = \left[m({}^A_Z X_N) - m({}_{Z-1}^{\ \ A}X'_{N+1}) - 2m_e\right] c^2 + \left[\sum_{i=1}^{Z} B_i - \sum_{i=1}^{Z-1} B'_i\right] \ . \tag{15.12}$$

Here we note that the electron rest-mass energies do not completely cancel. However, the discussion regarding the electron binding energy remains the same.

Q for electron capture

For electron capture:

$$Q_\varepsilon = \left[m({}^A_Z X_N) - m({}_{Z-1}^{\ \ A}X'_{N+1})\right]c^2 - B_n + \left[\sum_{i=1}^{Z} B_i - \sum_{i=1}^{Z-1} B'_i\right] . \qquad (15.13)$$

The latter term related to electron binding energy,

$$\sum_{i=1}^{Z} B_i - \sum_{i=1}^{Z-1} B'_i$$

is generally ignored, for the reasons cited above. However, the the binding energy of the captured electron, B_n can approach 100 keV for large-Z nuclei, and can *not* be ignored.

Discussion point: Free neutron decay, revisited

From our current understanding of the weak interaction, the electron is created when a down quark changes into an up quark. The Q value for this reaction is 0.782 MeV. Let us see if we can apply some reasoning from classical physics to say something about the observation of such a decay.

If the electron were a "point" particle, and it was created somewhere inside the neutron at radius r, is would feel an attraction:

$$V(r) = -\frac{e^2}{4\pi\epsilon_0}\left\{\frac{\Theta(R_p - r)}{R_p}\left[\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R_p}\right)^2\right] + \frac{\Theta(r - R_p)}{r}\right\} ,$$

where R_p is the radius of the proton. We are assuming that the quarks are moving so fast inside the proton, that all the electron sees is a continuous blur of charge adding up to one unit of charge. So, if $R_p \approx 1.2$fm (from $R_{\mathrm{N}} = 1.22 - 1.25[\mathrm{fm}]A^{1/3}$), we can conclude that the kinetic energy that the electron is required to have to escaped the nucleus falls in the range:

$$\begin{aligned} \frac{e^2}{4\pi\epsilon_0 R_p} &\le T_e \le \frac{3}{2}\frac{e^2}{4\pi\epsilon_0 R_p} \\ 1.2[\mathrm{MeV}] &\le T_e \le 1.8[\mathrm{MeV}] \end{aligned}$$

in other words, it can not happen. This is in contradiction with the observation that it does decay, with a meanlife of about 11 minutes.

Class discussion: Can you explain this?

15.2 Fermi's theory of β decay

Fermi's theory of β decay starts with a statement of Fermi's Golden Rule #2 for transition rate, λ:

$$\lambda = \frac{2\pi}{\hbar}|V_{if}|^2\rho(E_{if}) \ , \tag{15.14}$$

where V is a potential that causes the transition from an initial quantum state Ψ_i (the parent nucleus in the this case) to a final one, Ψ_f, that includes wavefunctions of the daughter nucleus, the electron and its neutrino. $V_{if} \equiv \langle\Psi_f|V|\Psi_i\rangle$ is the transition amplitude.

The derivation of Fermi's Golden Rule #2 is generally reserved for graduate courses in Quantum Mechanics, but a version of the derivation is available in Chapter 13, for your interest.

What concerns us now, is to calculate the density of final states, $\rho(E_{if})$, for the β-transition. This derivation figures so prominently in the β-spectrum, and the endpoint energy.

Starting in Chapter 13, the density of states is derived for non-relativistic particles with mass, relativistic particles with mass (the electron in this case), and massless particles (the neutrino in this case).

We start with (13.21). The number of states, N, of a particle in the final state with energy E is given by:

$$\mathrm{d}N = \frac{\pi}{2}n^2\mathrm{d}n \ . \tag{15.15}$$

where $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$, and (n_x, n_y, n_z) are the quantum numbers of a free particle in n infinite box potential, with side L. the momentum and the n's are related by:

$$p_i = n_i\pi\hbar/L \ . \tag{15.16}$$

Putting (15.16) into (15.15) gives:

$$\mathrm{d}N = \frac{1}{2\pi^2}\frac{L^3}{\hbar^3}p^2\mathrm{d}p \ . \tag{15.17}$$

Or, dividing by dE,

$$\frac{\mathrm{d}N}{\mathrm{d}E} = \frac{1}{2\pi^2}\frac{L^3}{\hbar^3}p^2\frac{\mathrm{d}p}{\mathrm{d}E} \ . \tag{15.18}$$

We should point out that (15.18) is valid for all particles, massless, relativistic and non-relativistic, since (15.16) is universal.

All we need do now is relate momentum to energy to compute the density factors. For the neutrino, which we are now treating as massless,

$$\begin{aligned} p_\nu &= E_\nu/c \\ \mathrm{d}p_\nu &= \mathrm{d}E_\nu/c \\ \frac{\mathrm{d}N_\nu}{\mathrm{d}E_\nu} &= \frac{1}{2\pi^2}\frac{L^3}{\hbar^3c^3}E_\nu^2 \end{aligned} \tag{15.19}$$

For the electron, that must be treated relativistically,

$$\begin{aligned} p_e &= \sqrt{E_e^2-(m_ec^2)^2}/c \\ \mathrm{d}p_e &= [E_e/(c\sqrt{E_e^2-(m_ec^2)^2})]\mathrm{d}E_e \\ \frac{\mathrm{d}N_e}{\mathrm{d}E_e} &= \frac{1}{2\pi^2}\frac{L^3}{\hbar^3c^3}\sqrt{E_e^2-(m_ec^2)^2}E_e \\ \frac{\mathrm{d}N_e}{\mathrm{d}T_e} &= \frac{1}{2\pi^2}\frac{L^3}{\hbar^3c^3}\sqrt{T_e(T_e+2m_ec^2)}(T_e+m_ec^2) \end{aligned} \tag{15.20}$$

For β decay we have two particles in the final state, so we can express the rate of decay to produce an electron with momentum p as:

$$\frac{\mathrm{d}\lambda_\beta}{\mathrm{d}p} = \frac{2\pi}{\hbar}|V_{if}|^2\frac{\mathrm{d}N_e}{\mathrm{d}p}\frac{\mathrm{d}N_\nu}{\mathrm{d}E_{if}} \ , \tag{15.21}$$

If q is the momentum of the neutrino,

$$\begin{aligned} E_{if} &= T_e + cq \\ \mathrm{d}E_{if} &= c(\mathrm{d}q) \quad (T_e \text{ fixed}) \ . \end{aligned} \tag{15.22}$$

Thus,

$$\frac{\mathrm{d}\lambda_\beta}{\mathrm{d}p} = \frac{2\pi}{\hbar c}|V_{if}|^2\frac{1}{2\pi^2}\frac{L^3}{\hbar^3}p^2\frac{1}{2\pi^2}\frac{L^3}{\hbar^3}q^2\delta(E_{if}-[T_e+T_\nu]) \ . \tag{15.23}$$

Where the δ-function accounts specifically for the conservation of energy.

Recall that the free electron and neutrino wavefunctions have the form $L^{-3/2}\exp(i\vec{p}\cdot x/\hbar)$ and $L^{-3/2}\exp(i\vec{q}\cdot x/\hbar)$, respectively. Thus, the L for the side of the box disappears from the calculation. We also replace $q = (Q - T_e)/c$, ignoring the recoil of the daughter nucleus. Finally, integrating over all possible neutrino energies, we obtain:

$$\begin{aligned} \frac{\mathrm{d}\lambda_\beta}{\mathrm{d}p} &= \frac{|\mathcal{M}_{if}|^2}{2\pi^3\hbar^7c^3}p^2(Q-T_e)^2 \text{ or} \\ \frac{\mathrm{d}\lambda_\beta}{\mathrm{d}p} &= \frac{|\mathcal{M}_{if}|^2}{2\pi^3\hbar^7c}p^2q^2 \end{aligned} \tag{15.24}$$

where $\mathcal{M}_{if} = L^3V_{if}$.

Thus we have derived Fermi's celebrated equation.

Just a brief note on dimensions: $|V_{if}|^2$ has units $[E^2]$ because all the wavefunctions inside are normalized. Getting rid of all the L's results in $\mathcal{M}_{if}$ having units [length$^3\times$ energy]. (15.24) *is* correct dimensionally.

Allowed transitions

Now we examine the form of the "matrix element" $\mathcal{M}_{if}$. This has changed form several times during the derivation, and will again, to conform with Krane's book.

We now rewrite

$$\begin{aligned} \mathcal{M}_{if} &= gM_{if} \\ M_{if} &= \langle (e^{i\vec{p}\cdot x/\hbar})(e^{i\vec{q}\cdot x/\hbar})\psi_{X'}|\mathcal{O}_\beta|\psi_X\rangle \ , \end{aligned} \tag{15.25}$$

where g is the "strength" of the β transition. This is a scalar quantity that plays the role of e, the electric charge, for electromagnetic transitions. The unnormalized electron wavefunction is $\exp(i\vec{p}\cdot x/\hbar)$, and the unnormalized neutrino wavefunction is $\exp(i\vec{q}\cdot x/\hbar)$. $\psi_{X'}$ is the wavefunction of the daughter nucleus, while ψ_X is the wavefunction of the parent nucleus. Finally, $\mathcal{O}_\beta$ is the weak interaction operator, the cause of the transition.

We recall from the class discussions, that the electron and neutrino wavefunctions have wavelengths that are many times the size of the nucleus. So, it seems reasonable to expand these wavefunctions in a Taylor series expansion, to see how far we get. Namely,

$$\exp(i\vec{p}\cdot x/\hbar) = 1 + \frac{i\vec{p}\cdot x}{\hbar} - \left(\frac{\vec{p}\cdot x}{\hbar}\right)^2 + \cdots$$

$$\exp(i\vec{q}\cdot x/\hbar) = 1 + \frac{i\vec{q}\cdot x}{\hbar} - \left(\frac{\vec{q}\cdot x}{\hbar}\right)^2 + \cdots \tag{15.26}$$

Thus the leading-order term of (15.25) is:

$$M_{if}^0 = \langle\psi_{X'}|\mathcal{O}_\beta|\psi_X\rangle \ . \tag{15.27}$$

If $M_{if}^0 \neq 0$, the β decay is called an "allowed" transition, and the rate is relatively prompt. If $M_{if}^0 = 0$, then we must go to higher order terms in (15.26). These are called "forbidden" transitions, and occur, but at much slower rates. (More on this topic later.)

Krane likes to adopt the following shorthand. For allowed transitions, we see that:

$$\frac{\mathrm{d}\lambda_\beta^0}{\mathrm{d}p} = g^2\frac{|M_{if}^0|^2}{2\pi^3\hbar^7c}p^2q^2 \ . \tag{15.28}$$

If we have $N(t)$ β-emitters in a sample, the momentum spectrum of electrons that may be measured is:

$$N^0(p)\mathrm{d}p = N(t)\mathrm{d}\lambda_\beta^0 = \left(g^2N(t)\frac{|M_{if}^0|^2}{2\pi^3\hbar^7c^5}\right)p^2q^2\mathrm{d}p \ . \tag{15.29}$$

If $N(t)$ changes little over the course of the measurement of the spectrum (the usual case):

$$N^0(p)\mathrm{d}p = C^{(0)}p^2q^2\mathrm{d}p \ , \tag{15.30}$$

where we have gathered all constants with inside the large parentheses in (15.29) into a global constant $C^{(0)}$, that is determined experimentally. It can be determined through the a normalization condition,

$$\int \mathrm{d}p\, N^0(p) \equiv 1 \ .$$

Conventional forms: $N^0(p)$, $N^0(T_e)$

$N^0(p)$ expressed in (15.30) contains p and q, that are related by conservation of energy. In terms of single momentum variable,

$$N^0(p)\mathrm{d}p = \frac{C^{(0)}}{c^2}p^2\left[Q - \sqrt{(cp)^2 + (m_ec^2)^2} + m_ec^2\right]^2\mathrm{d}p \ , \tag{15.31}$$

using relativistic kinematic relationships. The maximum possible p occurs when the neutrino component drops to zero. This is easily found to be:

$$p_{\max} = \frac{1}{c}\sqrt{Q^2 + 2Qm_e c^2} \ . \tag{15.32}$$

An even more common expression is to show N^0 in terms of T_e.

We find this by saying:

$$N^0(T_e)\mathrm{d}T_e = N^0(p)\mathrm{d}p = N^0(p)\left(\frac{\mathrm{d}p}{\mathrm{d}T_e}\right)\mathrm{d}T_e \ , \tag{15.33}$$

Applying relativistic kinematic relationships, we find:

$$N^0(T_e)\mathrm{d}T_e = \frac{C^{(0)}}{c^5}\sqrt{T_e^2 + 2T_e m_e c^2}(T_e + m_e c^2)(Q - T_e)^2\mathrm{d}T_e \ . \tag{15.34}$$

Here the β-endpoint at $Q = T_e$ is evident.

Accounting of "forbiddeness" and nuclear Coulomb effect.

There are two other attributes of β-spectra we must take account of, before we start using the theoretical spectral shape to assist in analyzing data.

The first of these has to do with the interaction of the daughter's Coulomb charge with the resultant electron or positron in the final state. This nuclear charge has no effect, of course, on the neutral neutrino. Going back to (15.25), we wrote the electron wavefunction as a free plane wave. In actual fact, that was a fairly crude approximation. These plane waves are distorted significantly by the attraction the β^- would feel, and the repulsion that the positron would feel. Incidentally, there is no effect on our conclusions regarding "allowed" or "forbidden".

Accounting for this is quite involved, but not beyond our capabilities. We would have to go back to (15.25) and write the electron wave functions in terms of free particle solutions to the Coulomb potential. (In NERS 311 we learn a lot about bound states of the Coulomb potential.) I have never seen detailed discussion of this in even graduate-level texts, and interested students are usually told to seek out the papers in the literature. The result is, however, that the β-spectra are multiplied by a correction factor, the *Fermi function*, that depends on the charge of the daughter nucleus, Z', and the electron momentum and sign, $F^{\pm}(Z', p)$. The effect it has could have been anticipated from classical considerations. The electron spectra is dragged back toward lesser values, while the positron spectra are pushed toward higher values. See Figure (9.3) in Krane.

The "forbiddeness" of the decay also affects the shape of the spectrum. This is also a multiplicative correction to the β-spectrum. There are difference shapes depending on the level of "forbiddeness", and that is determined by the amount of orbital angular momentum, L, carried away by the electron-neutrino pair, as well as their momenta. Examples of these shape factors are given in Table 15.2, for the "unique forbidden transitions"[3].

L	$S^L(p,q)$	
0	1	Allowed
1	$(p^2+q^2)/(m_ec)^2$	Unique first forbidden
2	$(p^4+\frac{10}{3}p^2q^2+q^4)/(m_ec)^4$	Unique second forbidden
3	$(p^6+7p^4q^2+7p^2q^4+q^6)/(m_ec)^6$	Unique third forbidden
$\vdots$	$\vdots$	$\vdots$

Table 15.1: Shape factors for the first three unique forbidden transitions.

The β-spectrum revealed

With all these various factors affecting the spectral shape and decay rates for β decay, we write down the final form that is employed for data analysis:

$$N(p) \propto |M_{if}^L|^2 p^2 (Q-T_e)^2 S^L(p,q) F^{\pm}(Z',p) \ , \tag{15.35}$$

where,

1. M_{if}^L is the nuclear matrix element associate with the transition. It can depend on p and q, as well as the alignment of spin and angular momentum vectors. It exhibits a very strong dependence on the angular momentum, L, carried off by the lepton pair. M_{if}^L also depends strongly on the "closeness" of the initial and final nuclear quantum wavefunctions. The closer the initial and final nuclear quantum states are, the larger their overlap, resulting in a larger M_{if}^L.

2. $p^2(Q-T_e)^2$ is the "statistical factor" associated with the density of final states.

3. $F^{\pm}(Z',p)$, the Fermi function. It accounts for the distortion of the spectral shape due to attraction/repulsion of the electron/positron.

4. $S^L(p,q)$ accounts for spectral shape differences. It depends on the total orbital angular momentum carried off by the electron-neutrino pair, $\vec{L}$, their total spin value, $\vec{S}$, and their orientation with respect to each other.

[3]Relativistic quantum mechanics allows us to calculate these in the special case of unique transitions. These transitions are ones in which the angular momentum vector and the two lepton spins are all aligned.

15.3 Experimental tests of Fermi's theory

Kurie plots: Shape of the β spectrum

To employ (15.35) to analyze β spectra, one plots:

$$\sqrt{\frac{N(p)}{S^L(p,q)F^{\pm}(Z',p)}} \quad vs. \quad T_e, \tag{15.36}$$

using the initial assumption that $L = 0$, so that $S^L(p,q) = 1$. If the data points fall an a straight line (statistical tests may be necessary), once can easily obtain the Q-value from the x-intercept. This type of plot is called a Kurie plot (named after Franz Kurie.) one has also identified, from the shape, that this is an allowed transition.

If the Kurie plot is not straight, one must successively test shape factors until a straight line match is obtained. Once the shape factor is determined, the level of forbiddeness is determined, and the Q-value may be extrapolated from the data unambiguously.

Total decay rate: The $ft_{1/2}$, $\log_{10} ft$ values

Putting in the Coulomb and shape factors into (15.28) allows us to determine the total decay rate for a β-decay process,

$$\begin{aligned}
\lambda_\beta &= g^2\frac{|M_{if}^L|^2}{2\pi^3\hbar^7 c}\int_0^{p_{\max}} \mathrm{d}p\, S^L(p,q)F^{\pm}(Z',p)p^2q^2 \\
&= g^2\frac{m_e^5c^4|M_{if}^L|^2}{2\pi^3\hbar^7}\left[\frac{1}{(m_ec)^5}\int_0^{p_{\max}} \mathrm{d}p\, S^L(p,q)F^{\pm}(Z',p)p^2q^2\right] \\
&\equiv g^2\frac{m_e^5c^4|M_{if}^L|^2}{2\pi^3\hbar^7}f_L(Z',Q)\ ,
\end{aligned} \tag{15.37}$$

where the dimensionless integral in large square brackets, is a theoretical factor that may be pre-computed and employed in the data analysis. This is conventionally written in terms of halflife, $t_{1/2} = \log(2)/\lambda_\beta$.

Thus,

$$f_L(Z',Q)t_{1/2} \equiv ft_{1/2} = \frac{\log_e(2)2\pi^3\hbar^7}{g^2m_e^5c^4|M_{if}^L|^2}\ . \tag{15.38}$$

This is known colloquially as the ft value. (Pronounced *eff tee*.) The ft's can be quite large, and sometimes the "log ft" value is quoted. (Pronounced *log eff tee*.) The precise definition is $\log_{10}(ft_{1/2})$.

Mass of the neutrino

Our applications of β-decay ignore the neutrino mass, but they turn out to be critically important for cosmology.

There is one important fact: they **do** have mass, but it is very small.

The table below shows the current state of the mass determinations of the three generations of leptons, e, μ, and τ.

lepton flavor	neutrino symbol	mass (eV)
e	ν_e	$0.04 \longrightarrow 2.2$
μ	ν_μ	$< 1.70 \times 10^5$
τ	ν_τ	$< 1.55 \times 10^7$

15.4 Angular momentum and parity selection rules

Classification of transitions in β decay

The e and the ν in the final states of a β decay each have intrinsic spin-$\frac{1}{2}$. Conservation of total angular momentum requires that:

$$\vec{I}_X = \vec{I}_{X'} + \vec{L} + \vec{S} \ , \tag{15.39}$$

where $\vec{I}_X$, $\vec{I}_{X'}$ are the total angular momenta of the parent and daughter, respectively, and $\vec{L}$, $\vec{S}$ are the total orbital and total spin angular momentum, respectively, of the $e\nu$ pair.

Therefore, the ΔI can be $\pm L$. or $\pm|L\pm1|$. If $L = 0$, then $\Delta I = \pm 1$. There are only two cases for lepton spin alignment. $S = 0$, when the $e\nu$ intrinsic spins anti-align, is called a Fermi transition. $S = 1$, when the $e\nu$ intrinsic spins align, is called a Gamow-Teller transition. Generally, as $L \uparrow$, $\lambda \downarrow$, $t_{1/2} \uparrow$, because there is much less overlap of the $e\nu$ wavefunctions with the nucleus.

The entire characterization scheme is given in Table 15.4 .

Nomenclature alert!

Type of Transition	Selection Rules	$L_{e\nu}$	$\Delta\pi$?	ft
superallowed	$\Delta I = 0, \pm 1^*$	0	no	1×10^3–1×10^4
allowed	$\Delta I = 0, \pm 1$	0	no	2×10^3–10^6
1^{st} forbidden	$\Delta I = 0, \pm 1$	1	yes	10^6–10^8
unique$^{**}1^{\mathrm{st}}$ forbidden	$\Delta I = \pm 2$	1	yes	10^8–10^9
2^{nd} forbidden	$\Delta I = \pm 1^{***}, \pm 2$	2	no	2×10^{10}–2×10^{13}
unique 2^{nd} forbidden	$\Delta I = \pm 3$	2	no	10^{12}
3^{rd} forbidden	$\Delta I = \pm 2^{***}, \pm 3$	3	yes	10^{18}
unique 3^{rd} forbidden	$\Delta I = \pm 4$	3	yes	4×10^{15}
4^{th} forbidden	$\Delta I = \pm 3^{***}, \pm 4$	4	no	10^{23}
unique 4^{th} forbidden	$\Delta I = \pm 5$	4	no	10^{19}

Table 15.2: Classification of transitions in β decay. Notes: (*) $0^+ \to 0^+$ can only occur via Fermi decay. (**) Unique transitions are Gamow-Teller transitions where $\vec{L}$ and $\vec{S}$ are aligned. The shape factors have very simple forms in this case. (***) For the $n \geq 2$ forbidden transitions, the $\Delta I = \pm(n-1)$ transition is often associated with the $n-2$ forbidden transition, being indistinguishable in the measurements of these processes.

Nomenclature	Meaning
$\vec{L}, L$	Total orbital angular momentum of the $e\nu$ pair
$\vec{S}, S$	Total spin angular momentum of the $e\nu$ pair
Fermi (F) transition	$e\nu$ intrinsic spins anti-align, $S = 0$
Gamow-Teller (GT) transition	$e\nu$ intrinsic spins align, $S = 1$
Superallowed	The nucleon that changed form, did not change shell-model orbital.
Allowed	$L = 0$ transition. $M_{if}^0 \neq 0$. See (15.27).
n^{th} forbidden	The $e\nu$ pair carry off n units of orbital angular momentum
Unique	$\vec{L}$ and $\vec{S}$ are aligned.

Examples of allowed β decays

This is straight out of Krane.

$^{14}O(0^+) \rightarrow ^{14}N^*(0^+)$ must be a pure Fermi decay since it is $0^+ \rightarrow 0^+$. Other examples are ^{34}Cl$\rightarrow^{34}$S, and ^{10}C$\rightarrow^{10}$B*.

$^{6}He(0^+) \rightarrow ^{6}Li(1^+)$, a $0^+ \rightarrow 1^+$ transition. This must be a pure Gamow-Teller decay. Other similar examples are ^{13}B$(\frac{3}{2}^-)\rightarrow^{13}C(\frac{1}{2}^-)$, and ^{230}Pa$(2^-)\rightarrow^{230}Th^*(3^-)$.

$n(\frac{1}{2}^+) \rightarrow p(\frac{1}{2}^+)$ This is a mixed transition. The F transition preserves the nucleon spin direction, the GT transition flips the nucleon spin. (Show drawing.)

β decay can either be of the F type, the GT type or a mixture of both. We may generalize the matrix element and coupling constant as follows, for allowed decays:

$$gM^0 = g_{\rm F} M^0_{\rm F} + g_{\rm GT} M^0_{\rm GT} = g_{\rm F}\langle\psi_{X'}|\mathbf{1}|\psi_X\rangle + g_{\rm GT}\langle\psi_{X'}|\mathcal{O}_{\uparrow\downarrow}|\psi_X\rangle \ , \tag{15.40}$$

where $\mathcal{O}_{\uparrow\downarrow}$ symbolizes an operator that flips the nucleon spin for the GT transition. The operator for the F transition is simply $\mathbf{1}$, (*i.e.* unity), and just measures the overlap between the initial and final nuclear states.

The fraction of F transitions is:

$$f_{\rm F} = \frac{g^2_{\rm F}|M^0_{\rm F}|^2}{g^2_{\rm F}|M^0_{\rm F}|^2 + g^2_{\rm GT}|M^0_{\rm GT}|^2} = \frac{y^2}{1+y^2} \ , \tag{15.41}$$

where,

$$y \equiv \frac{g_{\rm F} M^0_{\rm F}}{g_{\rm GT} M^0_{\rm GT}} \ . \tag{15.42}$$

Tables of y values are given in Krane on page 290.

15.4.1 *Matrix elements for certain special cases*

This section is meant to explain several things given without explanation in Krane's Chapter 9.

$M_{if} = \sqrt{2}$, for superallowed $0^+ \to 0^+$ transitions

This was stated near the top of the text on Krane's p. 284.

We know that a $0^+ \to 0^+$ allowed transition (super or regular), must be an F transition. In the case that it is also a superallowed transition, we can write explicitly:

$$M_{if} = \left\langle \psi_{X'}(0^+) \left(\frac{1}{\sqrt{2}}[e(\uparrow)\nu(\downarrow) + e(\downarrow)\nu(\uparrow)] \right) \middle| \mathbf{1} \middle| \psi_X(0^+) \right\rangle , \tag{15.43}$$

where the intrinsic spins of the $e\nu$ pair are shown explicitly. This spin wavefunction is properly normalized with the $\sqrt{2}$ as shown.

Separating the spins part, and the space part,

$$M_{if} = \frac{1}{\sqrt{2}} \langle \psi_{X'} | \psi_X \rangle \langle (e(\uparrow)\nu(\downarrow) + e(\downarrow)\nu(\uparrow)) | \vec{0} \rangle \;= \sqrt{2} , \tag{15.44}$$

since $\langle \psi_{X'} | \psi_X \rangle = 1$ for superallowed transitions, and $\langle e(\uparrow)\nu(\downarrow) | \vec{0} \rangle = \langle e(\downarrow)\nu(\uparrow)) | \vec{0} \rangle = 1$.

Using this knowledge, one can measure directly, $g_{\rm F}$ from $0^+ \to 0^+$ superallowed transitions. Adapting (15.38) for superallowed transitions,

$$g_{\rm F}^2 = \frac{\log_e(2)\pi^3\hbar^7}{m_e^5 c^4} \left(\frac{1}{f t_{1/2}} \right)_{\rm meas} , \tag{15.45}$$

giving a direct measurement of $g_{\rm F}$ via measuring ft. Table 9.2 in Krane (page 285) shows how remarkable constant ft is for $0^+ \to 0^+$ superallowed transitions. This permits us to establish the value for $g_{\rm F}$ to be:

$$g_{\rm F} = 0.88 \times 10^{-4} \text{ MeV} \cdot \text{fm}^3 . \tag{15.46}$$

15.5 Comparative half-lives and forbidden decays

Not covered in NERS312.

15.6 Neutrino physics

Not covered in NERS312.

15.7 Double-β Decay

Not covered in NERS312.

15.8 β-delayed electron emission

Not covered in NERS312.

15.9 Non-conservation of parity

Not covered in NERS312.

15.10 β spectroscopy

Not covered in NERS312.

$M_{if} = 1$, for neutron β decay, $n \to p + e^- + \tilde{\nu}_e$

This was stated near the top of the text on Krane's p. 290.

In this case, for an F transition:

$$M_{if} = \left\langle \psi_{X'}(0^+) \left(\frac{1}{\sqrt{2}}[e(\uparrow)\nu(\downarrow) + e(\downarrow)\nu(\uparrow)] \right) \middle| \mathbf{1} \middle| \psi_X(0^+) \right\rangle , \tag{15.47}$$